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Generalized Ideals in Orthoalgebras

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The definitions of generalized ideal and generalized filter in orthoalgebras are given, the relationship between generalized ideals and local ideals is studied, and the connections between generalized ideals and supports are established.

KEY WORDS: orthoalgebras; local ideals; generalized ideals; supports.

1. INTRODUCTION AND BASIC DEFINITIONS

Since in 1936 Birkhoff and von Neumann regarded the lattice of all closed subspaces of a separable infinite-dimensional Hilbert space that is an orthomodular lattice as a proposition system for a quantum mechanical entity (Miklós, 1998), orthomodular lattices have been considered as a mathematical model for a calculus of quantum logic. With the development of the theory of quantum logics, orthoalgebras as a quantum structure that generalize orthomodular lattices, orthomodular posets, are also regarded as a mathematical model of quantum logic (Foulis *et al.*, 1992). Because quantum structures are all algebraic structures, their algebraic properties play an important role in studying the theory of quantum logic (Miklós, 1998). We know that the notion of ideals (i.e., p-ideals) is a very powerful tool to study quantum logic (Kalmbach, 1983). Hence, it is necessary to study ideals of orthoalgebras. From the point of logic, Foulis *et al.*, studied local filters, local ideals, and obtained some properties of local filters (Foulis *et al.*, 1992). In this note, we give the definitions of generalized ideals, generalized filters, prove the equivalence between generalized ideals and local ideals, and establish the relationship between generalized ideals and supports.

Definition 1.1. (Foulis *et al.*, 1992). An orthoalgebra (OA) is a set L containing two special elements 0, 1 and equipped with a partially defined binary operation

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$\oplus$ subject to the following conditions for all $p, q, r \in L$:

(i) (Commutativity) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$;
(ii) (Associativity) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$;
(iii) (Orthocomplementation) For any $p \in L$ there is a unique $q \in L$ such that $p \oplus q$ is defined, and $p \oplus q = 1$;
(iv) (Consistency) If $p \oplus p$ is defined, then $p = 0$.

If the assumptions of (ii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L.

Remark 1.2. Let L be an orthoalgebra and $p, q \in L$.

(i) p is orthogonal to q and write $p \perp q$ iff $p \oplus q$ is defined in L.
(ii) p is less than or equal to q and write $p \leq q$ iff there exists an element $r \in L$ such that $p \oplus r = q$.
(iii) p is the orthocomplement of q iff p is a unique element of L such that $p \oplus q = 1$, and it is written as $q^{\perp}$.

Definition 1.3. Let L be an OA, a triple subset $\{p, q, r\}$ of L is called a triple orthogonal set if $p \perp q$ and $p \oplus q \perp r$ hold.

Definition 1.4. (Eissa and Habil, 1994). An orthomodular poset (OMP) is an orthoalgebra L that satisfies the following conditions:
For $p, q \in L$, if $p \perp q$, then $p \vee q$ exists and $p \vee q = p \oplus q$.

Lemma 1.5. *(Foulis et al., 1992). Let $p, q, r \in L$ with $p \perp q$ and $(p \oplus q) \perp r$. Then any of the following is a Boolean subalgebra of L:*

(*i*) $\{0, 1, p, p^{\perp}\}$,
(*ii*) $\{0, 1, p, q, p \oplus q, p^{\perp}, q^{\perp}, (p \oplus q)^{\perp}\}$,
(*iii*) $\{0, 1, p, q, r, p \oplus q, p \oplus r, q \oplus r, p \oplus q \oplus r, p^{\perp}, q^{\perp}, r^{\perp}, (p \oplus q)^{\perp}, (q \oplus r)^{\perp}, (p \oplus r)^{\perp}, (p \oplus q \oplus r)^{\perp}\}$.

Proposition 1.6. *(Foulis et al., 1992). Let L be an OA, for $p, q \in L$, $p \leq q$, define $q - p = (p \oplus q^{\perp})^{\perp}$, then the following properties are satisfied:*

(*i*) *If $p \perp q$, then $p = (p \oplus q) - q$,*
(*ii*) *If $p \leq q$, then $q = p \oplus (q - p)$,*
(*iii*) *If $p \leq q \leq r$, then $(r - q) \oplus (q - p) = r - p$,*
(*iv*) *If $p \leq q \leq r$, then $(r - p) - (q - p) = r - q$,*
(*v*) *If $p \leq q$ and $r \leq q - p$, then $(q - p) - r = (q - r) - p$.*

Definition 1.7. (Kalmbach, 1983). Let L be an OA. A nonempty subset I of L is called an ideal, if

(i) For all $a, b \in I$, there exists $c \in I$ such that $a \leq c, b \leq c$,
(ii) I is a down set, that is to say, $a \in L, b \in I$, and $a \leq b$ imply $a \in I$.

Definition 1.8. (Foulis *et al.*, 1992). Let L be an OA. A finite set $D \subseteq L$ is called a difference set if either D is empty or there exists a strictly increasing sequence

$$P_0 < P_1 < P_2 < \cdots < P_{n-1} < p_n.$$

in L such that $D = \{p_k - p_{k-1} \mid k = 1, 2, \ldots, n\}$.

In addition, we define $\oplus D = p_n - p_0$.

Definition 1.9. (Foulis *et al.*, 1992). Let L be an OA, a subset $S \subseteq L$ is called a support if it satisfies the following conditions:

(i) $0 \notin S$;
(ii) For $p, q \in L$, $p \perp q$, $p \oplus q \in S \Leftrightarrow \{p, q\} \cap S \neq \emptyset$.

Lemma 1.10. *(Foulis et al., 1992). Let L be an* OA, $0 \notin S \subseteq L$, *the following conditions are equivalent:*

(i) *S is a support;*
(ii) *For any difference D, $\oplus D \in S \Leftrightarrow D \cap S \neq \emptyset$.*

2. GENERALIZED IDEALS

Definition 2.1. Let L be an OA. A nonempty subset I of L is called a generalized ideal if it satisfies the following conditions:

(I1) For $p \in L, q \in I$, and $p \leq q$ imply $p \in I$ (down set).
(I2) For all triple orthogonal set $\{p, q, r\}$, if $p \oplus r \in I, q \oplus r \in I$, then $p \oplus q \oplus r \in I$.

Definition 2.2. Let L be an OA. A nonempty subset F of L is called a generalized filter if it satisfies the following conditions:

(F1) For $p \in F, q \in L$, $p \leq q$ imply $q \in F$ (up set).
(F2) For all triple orthogonal set $\{p, q, r\}$, if $p \oplus r \in F$ and $q \oplus r \in F$, then $r \in F$.

The set of all generalized ideals (generalized filters) of L is denoted by $I(L)$ (by $\mathcal{F}(L)$). Each of these, with the empty set added, is a complete lattice.

A generalized ideal or generalized filter is called proper if it does not coincide with L. It is very easy to show that

(i) A generalized ideal I of an OA is proper iff $1 \notin I$.
(ii) A generalized filter F of an OA is proper iff $0 \notin F$.
(iii) $\{0\}$ is the smallest generalized ideal. $\{1\}$ is the smallest generalized filter.

Proposition 2.3. *Let L be an OA. If $I(F)$ is a proper generalized ideal (filter) and $p \in I(F)$, then $p^{\perp} \notin I(F)$.*

Proposition 2.4. *Let L be an OA. If I is a proper generalized ideal, and F is a proper generalized filter, then $I^{\perp} = \{p^{\perp} : p \in I\}$ is a proper generalized filter, $F^{\perp} = \{p^{\perp} : p \in F\}$ is a proper generalized ideals.*

Proof: Let $p \in I^{\perp}$, $q \in L$ and $p \le q$. Then $p^{\perp} \in I$ and $q^{\perp} \le p^{\perp}$. So $q^{\perp} \in I$ and $q = (q^{\perp})^{\perp} \in I^{\perp}$. Let $\{p, q, r\}$ be a triple orthogonal set, $p \oplus r \in I^{\perp}$, $q \oplus r \in I^{\perp}$. Then $(p \oplus r)^{\perp} \in I$ and $(q \oplus r)^{\perp} \in I$, that is to say $(1 - (p \oplus r)) \in I$, $(1 - (q \oplus r))I$, For $(1 - (p \oplus r)) \in I$, by $p \oplus r \le q^{\perp} \le 1$, then

$$\begin{aligned} 1 - (p \oplus r) &= (1 - q^{\perp}) \oplus (q^{\perp} - (p \oplus r)) = q \oplus (q^{\perp} - (p \oplus r)) \\ &= q \oplus ((1 - q) - (p \oplus r)) = q \oplus (1 - (p \oplus r \oplus q)) \in I. \end{aligned}$$

Similarly, $1 - (q \oplus r) = p \oplus (1 - (p \oplus r \oplus q)) \in I$. Evidently, $\{p, q, 1 - (p \oplus q \oplus r)\}$ is a triple orthogonal set. Then $p \oplus q \oplus (1 - (p \oplus q \oplus r)) = p \oplus q \oplus ((1 - r) - (p \oplus q)) = 1 - r \in I$, i.e., $r \in I^{\perp}$. Since I is a proper generalized ideal, then $1 \notin I$, i.e., $0 \notin I^{\perp}$. Hence $I^{\perp}$ is a proper generalized filter.

Similarly, we can prove $F^{\perp}$ is a proper generalized ideal. □

Proposition 2.5. *Let L be an orthomodular poset. For $a \in L$ and $a \neq 1$, then $[0, a] = \{q \in L : 0 \le q \le a\}$ is a proper generalized ideal.*

Proof: Obviously, (I1) is satisfied. Let $\{p, q, r\}$ be a triple orthogonal set with $p \oplus r \le a$, $q \oplus r \le a$. Then $p \vee r \le a$, $q \le q \vee r \le a$. By $p \vee r = p \oplus r \perp q$, then $(p \vee r) \oplus q = p \vee r \vee q$. Hence $p \vee q \vee r \le a$, i.e., $p \oplus q \oplus r \le a$. So (I2) is satisfied and $[0, a]$ is a proper generalized ideal. □

Definition 2.6. Let L be an OA, a nonempty subset $I \subseteq L$ is called a local ideal iff for all Boolean subalgebra $B \subseteq L$, $I \cap B$ is an ideal of B.

Theorem 2.7. *Let L be an OA, a nonempty subset $I \subseteq L$ is a local ideal iff I is a generalized ideal.*

Proof: "Only if" part. Let $x \le y$, $y \in I$. Then $x \perp y^{\perp}$. By Lemma 1.5 we have that $B = \{0, 1, x, y^{\perp}, x \oplus y^{\perp}, x^{\perp}, y, (x \oplus y)^{\perp}\}$ is a Boolean subalgebra of L, which implies $I \cap B$ is an ideal of B. By $y \in I$, $y \in B$, then $\downarrow y \cap B \subseteq I \cap B$. So $x \in \downarrow y \cap B \subseteq I \cap B$. i.e., $x \in I$. Hence, (I1) is satisfied. For all triple orthogonal set $\{p, q, r\}$, if $p \oplus r \in I, q \oplus r \in I$, then $B = \{0, 1, p, q, r, p \oplus q, p \oplus r, q \oplus r, p \oplus q \oplus r, p^{\perp}, q^{\perp}, r^{\perp}, (p \oplus q)^{\perp}, (p \oplus r)^{\perp}, (q \oplus r)^{\perp}, (p \oplus q \oplus r)^{\perp}\}$ is a Boolean subalgebra of L by Lemma 1.5. Obviously, $p \oplus r, q \oplus r \in I \cap B \subseteq B$. Since B is a Boolean subalgebra and $I \cap B$ is an ideal of B, we know that $(p \oplus r) \vee (q \oplus r) \in I \cap B \subseteq I$. Then $(p \oplus r) \vee (q \oplus r) = (p \vee r) \vee (q \vee r) = (p \vee q) \vee r$. For $(p \oplus q) \perp r$, then $p \oplus q \oplus r = (p \vee q) \vee r$. Therefore, $p \oplus q \oplus r = (p \oplus r) \vee (q \oplus r) \in I$. i.e., (I2) is satisfied. So I is a generalized ideal of L.

"If" part. For all Boolean subalgebra B, if $x \le y$, $x \in B$ and $y \in I \cap B$, by (I1), then $x \in I \cap B$. For all $x, y \in I \cap B$, since $\{x, y\} \subseteq B$ is compatible when B is a Boolean subalgebra, then there exists a triple orthogonal set $\{x_1, y_1, d\}$ such that $x = x_1 \oplus d$, $y = y_1 \oplus d$. Evidently, $x_1 \oplus y_1 \oplus d \in I \cap B$, $x \le x_1 \oplus y_1 \oplus d$, and $y \le x_1 \oplus y_1 \oplus d$. Then $x \vee y \le x_1 \oplus y_1 \oplus d$ which implies $x \vee y \in I \cap B$. So $I \cap B$ is an ideal of B. Hence, I is a local ideal. □

Remark 2.8.

(i) When L is a Boolean algebra, then the notions of ideals, local ideals, and generalized ideals are the same thing.

(ii) When L is an orthomodular lattice, any ideal is a generalized ideal (local ideal). Conversely, it is not true, a counterexample is given Fig. 1.

Fig. 1 is an orthomodular lattice which is not a Boolean algebra.

Let $I = \{c^{\perp}, e^{\perp}, b, d, f, 0\}$. Obviously, I is a generalized ideal, but I is not an ideal. For $b, f \in I$, there doesn't exist an element $g \in I$ such that $b \le g$, $f \le g$.

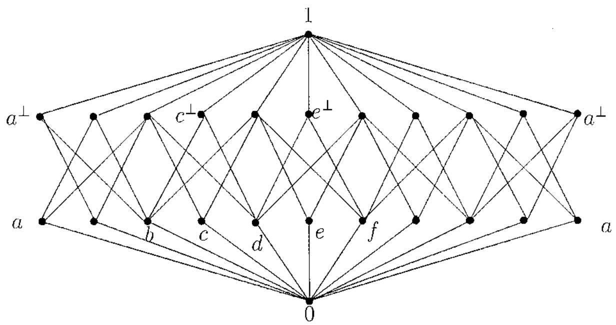

Fig. 1.

Theorem 2.9. *Let L be an OA. Then L is an orthomodular poset iff for all $a \in L$, $[0, a]$ is a generalized ideal.*

Proof: "Only if" part. By Proposition 2.5. "If" part., we only need to prove that for all $x, y \in L$, $x \perp y$ implies $x \vee y$ exists by Definition 1.4. For all $c \in L$, $x \leq c$, $y \leq c$, then $\{x, y, 0\}$ is a triple orthogonal set. By the assumption, then $x \oplus y \in [0, c]$, i.e., $x \oplus y \leq c$. So $x \oplus y = x \vee y$. □

Remark 2.10. Let L be an OA which satisfies the increasing chain condition. Then for all down set I, $I = \cup\{\downarrow p | p \in M\}$, where M is the set of maximal elements of I.

Proposition 2.11. *Let L be an OA which satisfies the increasing chain condition. I is a generalized ideal of L. p is a maximal element of I. Then for all $q \in I$, $q^{\perp} \vee p$ exists and $q^{\perp} \vee p = 1$.*

Proof: For all $r \in L$, if $p \leq r$, $q^{\perp} \leq r$, then $r^{\perp} \leq q$. Hence $r^{\perp} \in I$. Then $p \perp r^{\perp}$ by $p \leq r$. Obviously, $\{p, r^{\perp}, 0\}$ is a triple orthogonal set, which implies that $p \oplus 0 = p \in I$, $r^{\perp} \oplus 0 = r^{\perp} \in I$. Then $p \oplus r^{\perp} \oplus 0 \in I$. Since p is a maximal element, then $r^{\perp} = 0$. So $r = 1$. That is to say, $q^{\perp} \vee p = 1$. □

Corollary 2.12. *Let I be a generalized ideal and p, q be maximal elements of I. Then $p^{\perp} \vee q = 1$, $p^{\perp} \wedge q = 0$. (Therefore, maximal elements of a generalized ideal are perspective in the sense of* Kalmbach, 1983).

Let S be a support, and define $I_s = \{p \in L \mid p \notin S\}$.

Theorem 2.13. *Let S be a support of L, then I_s is a generalized ideal, and the assignment $S \mapsto I_s$ is an isomorphism from supports to generalized ideals.*

Proof: Obviously, if $S = \emptyset$, then $I_s = L$ is a generalized ideal of L. If S is a proper support, then S is an up set. Hence, I_s is a down set. For all triple orthogonal set $\{p, q, r\}$, if $p \oplus r \in I_s$, $q \oplus r \in I_s$, then $p \oplus r \notin S$, $q \oplus r \notin S$. So by Lemma 1.10 $p, q, r \notin S$. Again by Lemma 1.10, $p \oplus q \oplus r \notin S$. Hence $p \oplus q \oplus r \in I_s$. So I_s is a generalized ideal. In order to prove $S \mapsto I_s$ is an isomorphism we only have to prove for all generalized ideal I, there exists a support S such that $I = I_S$.

Let $S = \{p \in L \mid p \notin I\}$. For $0 \in I$, then $0 \notin S$. For $p, q \in L$, $p \perp q$, and $\{p, q\} \cap S \neq \emptyset$, suppose that $p \in S$, then $p \notin I$. So $p \oplus q \notin I$. Therefore $p \oplus q \in S$. Conversely, if $p \oplus q \in S$, then there is neither $p \in I$ nor $q \in I$. Otherwise, $\{p, q, 0\}$ is a triple orthogonal set, then $p \oplus q \in I$. So $\{p, q\} \cap S \neq \emptyset$. i.e., S is a support. Obviously $I = I_s$. □

Corollary 2.14. *Let L be an OA and I be a generalized ideal of L. Then $I^{\perp}$, $L \backslash I$ are a generalized filter, a support respectively, and $I^{\perp} \subseteq L \backslash I$.*

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